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Exterior solution for a charged radiating sphere in general relativity

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Abstract. The exterior solution for a radiating sphere in general relativity is known. We present here the complete exterior solution when the radiating sphere is charged.

1. Introduction

It has recently been suggested (Shvartsman 1971) that stars carry electrical charges, the sign and magnitude of which are determined by exchange processes between the star and the surrounding medium. Vaidya (1951) has worked out an exterior solution for a radiating star in general relativity. We investigate the same problem considering the spherical mass to be charged.

2. The field equations

A sphere of mass M, charge Q and radius r_0 is supposed to start radiating at time t_0. As the sphere continues to radiate the zone of radiation increases in thickness, its outer surface at a later instant t_1 being $r = r_1$. For $r_0 \leqslant r \leqslant r_1$, $t_0 \leqslant t \leqslant t_1$, let the line element be assumed to be of the form

$$ds^2 = -e^{\lambda}\, dr^2 - r^2(d\theta^2 - \sin^2\theta\, d\phi^2) + e^{\nu}\, dt^2 \tag{1}$$

where $\lambda = \lambda(r, t)$, $\nu = \nu(r, t)$. For the radiation we have an energy tensor $T^{\mu\nu}$ of the form

$$T^{\mu\nu} = \rho v^{\mu} v^{\nu} + E^{\mu\nu} \tag{2}$$

ρ is the density of radiation, $E^{\mu\nu}$ is the electromagnetic energy–momentum tensor, v^{μ} is $dx^{\mu}/d\tau$ with $d\tau = dx_0^1 = dx_0^4$ in the natural coordinate system. Since the lines of flow are null geodesics:

$$v_{\mu} v^{\mu} = 0, \qquad (v^{\mu})_{\nu} v^{\nu} = 0. \tag{3}$$

Since $(T^{\mu\nu})_{\nu} = 0$, we have the analogue of equation of continuity

$$(\rho v^{\mu})_{\mu} = 0. \tag{4}$$

As the flow is radial $v^2 = 0$, $v^3 = 0$.

$$\begin{aligned} &T_1^1 = \rho v_1 v^1 + \tfrac{1}{2}E, \qquad T_4^4 = \rho v_4 v^4 + \tfrac{1}{2}E, \qquad T_1^4 = \rho v_1 v^4 \\ &T_2^2 = T_3^3 = -\tfrac{1}{2}E \end{aligned} \tag{5}$$

where $E = Q^2/r^4$. Also $v_\mu v^\mu = 0$ simplifies to

$$-\mathrm{e}^{\lambda}(v^1)^2 + \mathrm{e}^{\nu}(v^4)^2 = 0. \tag{6}$$

With the usual expression for the components of $T^{\mu\nu}$ in terms of $g_{\mu\nu}$ and their derivatives, equation (5) gives the following field equations

(i)

$$8\pi T_1^4\,\mathrm{e}^{(\nu-\lambda)/2} + 8\pi T_4^4 = 4\pi E \tag{7}$$

or,

$$\mathrm{e}^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} + \frac{\dot{\lambda}}{r}\,\mathrm{e}^{-(\lambda+\nu)/2} = \frac{4\pi Q^2}{r^4} \tag{8}$$

(ii)

$$8\pi T_1^1 + 8\pi T_4^4 = 8\pi E \tag{9}$$

or,

$$\mathrm{e}^{-\lambda}\left(\frac{\lambda' - \nu'}{r} - \frac{2}{r^2}\right) + \frac{2}{r^2} = \frac{8\pi Q^2}{r^4} \tag{10}$$

(iii)

$$8\pi T_2^2 = 8\pi T_3^3 = -4\pi E \tag{11}$$

or,

$$\mathrm{e}^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu' - \lambda'}{2r}\right) + \mathrm{e}^{-\nu}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda}\dot{\nu}}{4}\right) = -\frac{4\pi Q^2}{r^4}. \tag{12}$$

Throughout, primes and superior dots indicate differentiation with respect to r and t, respectively.

If the total energy is to be conserved, the line element obtained by solving these equations must reduce to the static form

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{4\pi Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \left(1 - \frac{2M}{r} + \frac{4\pi Q^2}{r^2}\right)\mathrm{d}t^2 \tag{13}$$

at $r = r_0$, $t = t_0$ and for $r \geqslant r_1$ at $t = t_1$.

3. The solution of the field equations

On putting

$$\mathrm{e}^{-\lambda} = 1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}, \qquad m = m(r, t) \tag{14}$$

in the field equation (8) we find that it is equivalent to

$$\mathrm{e}^{-\lambda/2}\frac{\partial m}{\partial r} + \mathrm{e}^{-\nu/2}\frac{\partial m}{\partial t} = 0. \tag{15}$$

Using the operator

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = v^1 \frac{\partial}{\partial r} + v^4 \frac{\partial}{\partial t} \tag{16}$$

we may express this as

$$\frac{\mathrm{d}m}{\mathrm{d}\tau} = 0. \tag{17}$$

From equation (15) we can express $\mathrm{e}^{\nu/2}$ in terms of m

$$\mathrm{e}^{\nu/2} = -\frac{\dot{m}}{m'}\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)^{-1/2}. \tag{18}$$

Now we can take the second field equation (10). On substituting the value of λ and ν from equations (14) and (17), we find that

$$\left(\frac{\dot{m}'}{\dot{m}} - \frac{m''}{m'}\right)\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right) = \frac{2m}{r^2} - \frac{8\pi Q^2}{r^3}. \tag{19}$$

The first integral of the above equation is

$$m'\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right) = f(m) \tag{20}$$

$f(m)$ being an arbitrary function. Equation (20) is the differential equation to be solved for m.

The following is an identity holding between the components of tensor T^{ν}_{μ}

$$\frac{\partial}{\partial r}(T^1_1) + \frac{\partial}{\partial t}(T^4_1) - \frac{\nu'}{2}(T^4_4 - T^1_1) + \frac{2}{r}(T^1_1 - T^2_2) + T^4_1\left(\frac{\dot{\lambda} + \dot{\nu}}{2}\right) = 0. \tag{21}$$

With the help of this identity and the two equations (7) and (9) the equation (11) can be transformed into

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\mathrm{e}^{-\lambda}\left(r^2 T^4_4 - \frac{Q^2}{2r^2}\right)\right] = 0. \tag{22}$$

Thus the third field equation is satisfied, ie $T^2_2 = -4\pi Q^2/r^4$, provided equation (22) is satisfied, ie provided

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[m'\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)\right] = 0, \tag{23}$$

that is, provided $\mathrm{d}m/\mathrm{d}\tau = 0$ when we use equation (20). The last relation has already been proved as equation (17) above.

Hence we have solved all the field equations and the final line element describing the radiation envelope of a charged sphere is

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) + \frac{\dot{m}^2}{f^2}\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)\mathrm{d}t^2 \tag{24}$$

with

$$m'\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right) = f(m), \qquad m = m(r, t)$$

for $r_0 \leqslant r \leqslant r_1, t_0 \leqslant t \leqslant t_1$. The value of $f(m)$ depends on the conditions in the interior of a star.

The surviving components of the energy tensor are

$$-T_1^1 = T_4^4 = \frac{m'}{4\pi r^2} + \frac{Q^2}{2r^4}, \qquad T_1^4 = \frac{m'^2}{4\pi \dot{m} r^2}, \qquad T_4^1 = -\frac{\dot{m}}{4\pi r^2}. \tag{25}$$

In our case the operator $d/d\tau$ when operating upon $m, v^1, (r^2\rho v^1), (r^2\rho v^1 v^1)$ and $(r^2\rho)$ gives similar results to the case of an uncharged radiating star, eg

$$\frac{dm}{d\tau} = 0 \tag{26}$$

$$\frac{dv^1}{d\tau} = 0 \tag{27}$$

$$\frac{d}{d\tau}(r^2\rho v^1) = 0. \tag{28}$$

Combining equations (27) and (28)

$$\frac{d}{d\tau}(r^2\rho v^1 v^1) = 0 \tag{29}$$

or,

$$\frac{d}{d\tau}[r^2(T^{11} - E^{11})] = 0 \tag{30}$$

and

$$\frac{d}{d\tau}(r^2\rho) = 0. \tag{31}$$

From equation (30) we get

$$T_2^2 = -\frac{4\pi Q^2}{r^4} \tag{32}$$

which was also seen from the field equations.

From the definition of the operator

$$\frac{d}{d\tau} \equiv v^1\frac{\partial}{\partial r} + v^4\frac{\partial}{\partial t},$$

it is clear that it differentiates following the lines of flow. Hence the relations (26), (27) and (28) show that m, v^1 and $r^2\rho$ are conserved along the lines of flow as in Vaidya's case.

The actual values of v^1 and v^4 may now be deduced. From equations (26) and (27) we have

$$\frac{\partial v^1}{\partial r} - \frac{m'}{\dot{m}}\frac{\partial v^1}{\partial t} = 0. \tag{33}$$

Hence

$$v^1 = \phi(m), \qquad v^4 = -\frac{m'}{\dot{m}}\phi(m). \tag{34}$$

$\phi(m)$ is now to be obtained by using any one of the equations

$$-T_1^1 = T_4^4 = \frac{m'}{4\pi r^2}+\frac{Q^2}{2r^4}, \qquad T_1^4 = \frac{m'^2}{4\pi \dot{m} r^2}, \qquad T_4^1 = -\frac{\dot{m}}{4\pi r^2}.$$

Thus $4\pi r^2 \rho v_4 v^1 = -\dot{m}$ (the last of the above equations) or,

$$[\phi(m)]^2 = \frac{m'}{4\pi r^2 \rho}\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)$$

or,

$$v^1 = \left(\frac{f(m)}{4\pi r^2 \rho}\right)^{1/2}, \qquad v^4 = -\frac{m'}{\dot{m}}\left(\frac{f(m)}{4\pi r^2 \rho}\right)^{1/2}. \tag{35}$$

Lastly we may now verify that the principle of conservation of energy holds. The line element (1) can be expressed in the form

$$ds^2 = -[(dx)^2+(dy)^2+(dz)^2]-\frac{e^{\lambda-1}}{r^2}(x\,dx+y\,dy+z\,dz)^2+e^{\nu}\,dt^2. \tag{36}$$

By using the well known formula by Tolman (1934) the energy content of equation (36) is found to be

$$E = \lim_{r\to\infty}\,[\tfrac{1}{2}r(e^{\lambda}-1)\,e^{(\nu-\lambda)/2}].$$

Hence for all distributions for which the line element (1) goes off continuously over some boundary to Schwarzschild's form, the principle of conservation

$$E = M$$

holds.

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